Math 120 Calculus I
First Test Answers
September 2015

Scale. 95–100 A+. 88–94 A. 83–87 A-. 79–82 B+. 74–78 B. 69–73 B-. 64–68 C+. 58–63 C. 48–57 C-. 43–47 D+. 33–41 D. Median 66.

Extra credit. There will be an extra credit "challenge test" you can take on line available all day Monday and Tuesday, Sept. 28–29, for 48 hours.

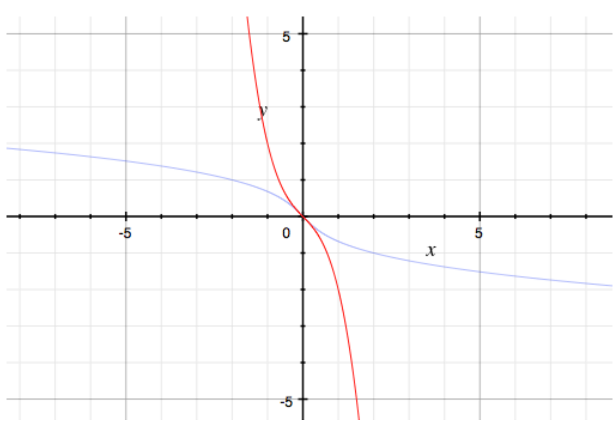
The challenge test will cover the same topics as the test with most of the questions coming from chapter 2. You'll find it on MyMathLab. You'll be able to do each question on it only once. You may use a calculator if you need it, but no book, notes, or internet. Do the test entirely by yourself. Do not get help from anyone or talk to anyone about the test until Wednesday after the test ends.

Whatever questions you do correctly on the challenge test will increase your grade for the first test. You can get up to half the difference between your test grade and 100. So, for example, if you got 70 on this test, a B-, you could increase your grade by 15 to 85, an A-, by getting all 20 problems correct on the challenge test, but if only 10 correct, then your grade would increase by 7.5 to 77.5, a B.

1. [8] Determine the exact value of $\arctan\sqrt{3}$. Express your answer as a multiple of π radians. Explain in a sentence or a diagram why your answer is correct.

Since the tangent of the acute angle 60° is $\sqrt{3}$, the arctangent of $\sqrt{3}$ is 60°, which in radians is $\pi/3$.

2. [8] The graph of $y = f(x) = -x^3 - x$ is shown below. On the same graph, sketch the graph of $y = f^{-1}(x)$.



The function f is graphed in red, its inverse f^{-1} is graphed in light blue.

3. [8] Note that both $\lim_{x\to0} x = 0$ and $\lim_{x\to0}(-x) = 0$. Determine the limit $\lim_{x\to0} x\sin(1/x)$ and explain how you made your determination.

There are two arguments you can give to show the limit is 0.

First proof: Since $-x \le x\sin(1/x) \le x$, and both $-x$ and x approach 0, therefore the function in between, $x\sin x$ also approaches 0 by the sandwich lemma.

Second proof: Since x approaches 0 and $\sin(1/x)$ is bounded between ±1, therefore they product also approaches 0.

An invalid argument is this: since x approaches 0, therefore x times anything else approaches 0. That's false in general; you can show that's false since x times $1/x$ approaches 1.

4. [32; 8 points each part] Evaluate the following limits. If a limit doesn't exist, then state briefly why not. (Use the properties of limits we've discussed so far in class. Later on this semester we'll discuss L'Hôpital's rule, but don't use that to determine these limits.)

a. $\lim_{x\to2} \dfrac{x^2+4}{x^2-4}$

This is of the form 8/0. It diverges.

b. $\lim_{x\to2} \dfrac{x^2-4}{x^2+3x+10}$

This is of the form 0/20. It converges to 0.

c. $\lim_{x\to0} \dfrac{3x}{\sin(4x)}$

The limit is 3/4. $\dfrac{3x}{\sin 4x} = \dfrac{3}{4}\dfrac{4x}{\sin 4x}$. Since the second fraction approches 1, the limit is equal to 3/4

d. $\lim_{x\to0} \dfrac{\dfrac{1}{2+x} - \dfrac{1}{2}}{x}$

The numerator $\dfrac{1}{2+x} - \dfrac{1}{2}$ simplifies to $\dfrac{-x}{2(2+x)}$, so the whole expression simplifies to $\dfrac{-1}{2(2+x)}$. As $x \to 0$, that approaches $-1/4$.

5. [14; 7 points each part] Consider the function $f(x) = 2x^3$.

a. Find the average rate of change of the f over the interval between 1 and b. Simplify your answer.

The average rate of change is

$$\frac{f(b)-f(1)}{b-1} = \frac{2b^3-2}{b-1}$$

The numerator factors as $2(b-1)(b^2+b+1)$, so the average rate of change simplifies to $2(b^2+b+1)$.

b. Use the information you found in part **a** to determine the instantaneous rate of change at $x = 1$.

Just take the limit as $b \to 1$ to get 6.

6. [8] On one-sided limits. Draw the graph of a function for which $\lim_{x \to 1^-} f(x) = 2$ but $\lim_{x \to 1^+} f(x) = 4$.

Your graph should display a 2-unit jump upward at $x = 1$. In order to be a function, there must be only one value at $x = 1$.

7. [14; 7 points each part] Consider the function $f(x) = 5x - 2$. In this problem, you'll use the definition to prove that the limit $\lim_{x \to 2} f(x)$ is equal to 8.

a. Let $\epsilon = 0.1$. Find a positive value of δ so that if x is within δ of 2, then $f(x)$ must be within $\epsilon = 0.1$ of 8. Show your work or explain in words why your value of δ will work.

We need to find a positive value of δ so that

$$|x - 2| < \delta \text{ implies } |f(x) - 8| < 0.1$$

Now, $f(x) - 8$ is equal to $5x - 10$, so the desired conclusion, $|f(x) - 8| < 0.1$ says $|5x - 10| < 0.1$, equivalently $|x - 2| < 0.02$. But we're looking for δ so that

$$|x - 2| < \delta \text{ implies } |x - 2| < 0.02$$

Clearly, $\delta = 0.02$ works.

b. Now show it for arbitrary positive ϵ. This is like part **a**, but all you know about ϵ is that it's positive. Your value of δ will depend on ϵ.

We're looking for δ so that

$$|x - 2| < \delta \text{ implies } |f(x) - 8| < \epsilon$$

With a small change in the computations above, the conclusion is equivalent to $|x - 2| < \epsilon/5$. So $\delta = \epsilon/5$ does the trick.

8. [8] On continuity. Consider the function

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 5 \\ 10x & \text{if } x < 5 \end{cases}$$

f is not continuous at $x = 5$. Explain in terms of limits why f is not continuous there.

The left limit is $\lim_{x \to 5^-} 10x = 50$ while the right limit is $\lim_{x \to 5^+} x^2 = 25$. Since the left and right limits are not equal, the limit as $x \to 5$ doesn't exist, so the function $f(x)$ is not continuous at 5.