CLARK UNIVERSITY 1887

# Math 121 Calculus II

Second Test Answers
March 2015

**Scale.** 98–100 A+, 93–97 A, 89–92 A–, 84–88 B+, 79–83 B, 74–78 B–, 70–73 C+, 60–69 C, 53–59 C, 45–52 D+, 35–44 D. Median 74. Students with C+ or lower should attend tutoring sessions.

**1.** [16] Evaluate the following indefinite integral.

$$\int \frac{\sqrt{x^2-4}}{x}\,dx$$

A trig sub works best here. Let $x = 2\sec\theta$ so that $\sqrt{x^2-4} = 2\tan\theta$ and $dx = 2\sec\theta\tan\theta\,d\theta$. Then the integral becomes

$$2\int \tan^2\theta\,d\theta$$

You can integrate that with the help of the Pythagorean identity $\sec^2\theta = 1+\tan^2\theta$, so the integral equals

$$2\int(\sec^2\theta - 1)\,d\theta = 2\tan\theta - \theta + C$$

Using the orignal substitution we can write that in terms of $x$ as

$$\sqrt{x^2-4} - \operatorname{arcsec}\tfrac{x}{2} + C$$

**2.** [24; 12 points each part] Evaluate the following indefinite integrals.

**a.** $\displaystyle\int x^2\sin x\,dx$

First use an integration by parts with $u = x^2$ and $dv = \sin x\,dx$ so that $du = 2x\,dx$ and $v = -\cos x$. Note that to find $v$ you have to integrate $\sin x$, not differentiate it. That converts the integral to

$$-x^2\cos x + \int 2x\cos x\,dx$$

To integrate the new integral, you'll need another integration by parts which results in the answer

$$-x^2\cos x + 2x\sin x + 2\cos x + C$$

**b.** $\displaystyle\int \cos^3 x\,\sin^2 x\,dx$

For this product of powers of trig functions, note that the power of $\cos x$ is odd, so the substitution $u = \sin x$ with $du = \cos x\,dx$ will work. First use the Pythagorean identity $\sin^2 x + \cos^2 x = 1$ to rewrite the integral as

$$\int(1-\sin^2 x)\cos x\,\sin^2 x\,dx$$

then make the substitution to finish the integration

$$\begin{aligned}\int(1-u^2)u^2\,du &= \int(u^2-u^4)\,du\\ &= \tfrac{1}{3}u^3 - \tfrac{1}{5}u^5 + C\\ &= \tfrac{1}{3}\sin^3 x - \tfrac{1}{5}\sin^5 x + C\end{aligned}$$

**3.** [12] Evaluate the following definite integral.

$$\int_1^e x\ln x\,dx$$

You can integrate

$$\int x\ln x\,dx$$

by parts with $u = \ln x$ and $dv = x\,dx$ so that $du = \frac{1}{x}\,dx$ and $v = \frac{1}{2}x^2$. (Note that you have to integrate $x$ to find $v$.) Then

$$\begin{aligned}\int x\ln x\,dx &= \tfrac{1}{2}x^2\ln x - \int \tfrac{1}{2}x\,dx\\ &= \tfrac{1}{2}x^2\ln x - \tfrac{1}{4}x^2\end{aligned}$$

Evaluate that between 1 and $e$ to get

$$(\tfrac{1}{2}e^2 - \tfrac{1}{4}e^2) - (0 - \tfrac{1}{4}) = \tfrac{1}{4}(e^2+1)$$

**4.** [12] Solve the separable differential equation

$$\frac{dy}{dx} = e^{-y} \cos x$$

Your answer should express $y$ as a function of $x$.

First separate the variables to get $e^y \, dy = \cos x \, dx$. Integrate that giving

$$\int e^y \, dy = \int \cos x \, dx$$

The integral of the left side is $e^y$, and the integral of the right side is $\sin x$, therefore

$$e^y = \sin x + C$$

where $C$ is an arbitrary constant. Solve that equation for $y$ to get

$$y = \ln(\sin x + C)$$

**5.** [20] Consider the rational function

$$7x + 7x^2 + 3x - 10$$

**a.** [4] Factor the denominator.

$$x^2 + 3x - 10 = (x+5)(x-2)$$

**b**. [8] Write the rational function as a sum of partial fractions.

It will look like

$$\frac{7x+7}{x^2+3x-10} = \frac{A}{x+5} + \frac{B}{x-2}$$

where you have to determine the constants $A$ and $B$. Do that by clearing the denominators.

$$7x + 7 = A(x-2) + B(x+5)$$

At this point there are various methods you can use to determine $A$ and $B$. One way is to set $x$ to convenient values such as the roots of the polynomial $x^2 + 3x - 10$. For $x = 2$, the equation says $21 = B7$, so $B = 3$. For $x = -5$, the equation says $-28 = A - 7$, so $A = 4$. We can now write the rational function as

$$\frac{7x+7}{x^2+3x-10} = \frac{4}{x+5} + \frac{3}{x-2}$$

**c.** [8] Use what you found in part **b** to evaluate this integral.

$$\int \frac{7x+7}{x^2+3x-10} \, dx$$

It is equal to

$$\int \left( \frac{4}{x+5} + \frac{3}{x-2} \right) dx = 4|\ln(x+5)| + 3|\ln(x-2)| + C$$

**6.** [16; 8 points each part] The most common form of radium, radium-226, has a half life of 1601 years.

**a.** Write down an formula that gives the amount $y$ of radium left after a period of $t$ years when the initial amount is $A$.

One formula that works is $y = A(\frac{1}{2})^{t/1601}$. You could also write it as $y = Ae^{kt}$ and determine $k$ by solving the equation $\frac{1}{2} = e^{1601k}$. You'll find $k = \dfrac{-\ln 2}{1601}$.

**b.** Use that formula to determine when only $\frac{1}{3}$ of the initial amount will remain. (Leave your answer in terms of exponents and logs.)

Solve $y = \frac{1}{3}A$ using your formula from part **a**. You'll find $t = 1601 \frac{\ln 3}{\ln 2}$ (which works out to be 2537.5)