Math 121 Calculus II

First Test Answers

February 2016

Scale. A+ 97–100. A 93–96. A- 89–92. B+ 84–88. B 79–83. B- 74–78. C+ 67–73. C 60–66. C- 53–59. D+ 44–52. D 36–43. Median 77.

1. [10] **On integrals and derivatives.** Suppose a function $f(x)$ defined on the interval $[3,7]$ has a maximum value of 5 and a minimum value of 2. Explain why the value of the integral $\int_3^7 f(x)\,dx$ cannot equal 6. (Write clearly and use full sentences. Suggestion: draw a figure.)

The integral $\int_3^7 f(x)\,dx$ is the area of the region under the curve $y=f(x)$ above the x-axis between $x=3$ and $x=7$. Since the function is greater than 2, this region includes the rectangle of height 2 above the interval $[3,7]$ on the x-axis. That 2 by 4 rectangle has area 8, therefore the integral of the function is greater than 8. It can't be as small as 6.

Another way to answer this question is to say that the average value on the interval $[3,7]$ lies between 2 and 5, but if the integral is 6, then its average value would be 6 divided by the length, 4, of the interval, giving 1.5, which is not between 2 and 5.

(Note the maximum being 5 says $f(x)$ can't be greater than 5, not that the integral $\int_3^7 f(x)\,dx$ can't be greater than 6. Indeed, the integral has to be greater than 8.)

2. [20; 10 points each part] **Evaluate the indefinite integrals.**

a. $\displaystyle\int \left(9x^5 - \sqrt{x} + \frac{7}{x}\right) dx$

$$\tfrac{3}{2}x^6 - \tfrac{2}{3}x^{3/2} + 7\ln|x| + C$$

b. $\displaystyle\int \sec^2(8x)\,dx$

One way to evaluate this integral is to make the substitution $u=8x$, $du = 8\,dx$. Then the integral becomes $\frac{1}{8}\int \sec^2 u\,du$, which equals $\frac{1}{8}\tan u + C = \frac{1}{8}\tan(8x)+C$.

3. [20; 10 points each part] **On definite integrals.** Evaluate the following integrals. Show your work for credit. (You do not have to find the answer decimally; an unsimplified expression involving numbers is sufficient.)

a. $\displaystyle\int_0^{\pi/2} (5\cos x - 2\sin x)\,dx$

By the fundamental theorem of calculus, this integral equals $5\sin x + 2\cos x\big|_0^{\pi/2} = (5\sin(\pi/2) + 2\cos(\pi/2)) - (5\sin 0 + 2\cos 0)$, which simplifies to $5-2=3$.

b. $\displaystyle\int_1^2 x^2 e^{x^3-1}\,dx$

A substitution with $u = x^3-1$ and $du = 3x^2\,dx$ will help. You'll get

$$\int_0^7 \tfrac{1}{3}e^u\,du = \tfrac{1}{3}e^u\Big|_0^7 = \tfrac{1}{3}(e^7-1)$$

4. [10; 5 points each part] **On areas between curves.** Consider the region in the plane between the straight line $y=\sqrt{5}-x$ and the hyperbola $y=1/x$.

a. The line and the hyperbola intersect at two points. What are their x-coordinates?

The points of intersection can be found by simultaneously solving the two equations $y=\sqrt{5}-x$ and $y=1/x$. Eliminating y gives the single equation $\sqrt{5}-x=1/x$, equivalent to the quadratic equation $x^2-\sqrt{5}\,x-1=0$ which has solutions $x=\frac{1}{2}(\sqrt{5}\pm 1)$. (By the way, the number $\frac{1}{2}(\sqrt{5}+1)$ is called the *golden ratio.*)

b. Write down an integral which gives the area of that region. Do not compute the value of that integral.

In the region between $x=\frac{1}{2}(\sqrt{5}-1)$ and $x=\frac{1}{2}(\sqrt{5}+1)$, the straight line is above the hyperbola. Therefore the integral is

$$\int_{(\sqrt{5}-1)/2}^{(\sqrt{5}+1)/2} \left(\sqrt{5}-x-\frac{1}{x}\right) dx$$

5. [20; 10 points each part] **On volumes of solids of revolution.** In each case write down an expression for the volume of this solid in terms of integrals. Don't valuate the integrals.

a. Consider the region below the curve $y=8\sin 2x$ and above the x-axis for x between 0 and $\pi/2$. Rotate this region around the x-axis to get a solid of revolution. Write down an integral which gives the volume of that solid of revolution. Do not compute the value of the integral.

You can use the disk method for this one.

$$\int_a^b \pi R(x)^2\,dx = \int_0^{\pi/2} \pi(8\sin 2x)^2\,dx$$

b. The solid of revolution is formed by rotating around the x-axis the region between the straight line $y=\sqrt{5}-x$ and the hyperbola $y=1/x$. (See the previous problem.) Write down an integral which gives the volume of that solid of revolution. Do not compute the value of the integral.

You found the limits of integration in the previous problem. They're $x = \frac{1}{2}(\sqrt{5} \pm 1)$.

For this one you'll need the washer method since the region being rotated doesn't go down to the x-axis.

$$\int_a^b (\pi R(x)^2 - \pi r(x)^2)\, dx = \int_{(\sqrt{5}-1)/2}^{(\sqrt{5}+1)/2} \left(\pi(\sqrt{5} - x)^2 - \frac{\pi}{x^2}\right) dx$$

6. [10] **On arc lengths.** Write down an integral which gives the length of that part of the parabola $x = y^2$ with endpoints $(4, -2)$ and $(4, 2)$. Do NOT evaluate the integral.

Since y is the independent variable in this question, you'll need to compute $\dfrac{dx}{dy}$, and the integral will be with respect to y instead of with respect to x. You'll find $\dfrac{dx}{dy} = 2y$, so

$$\int_a^b ds = \int_a^b \sqrt{1 + \left(\tfrac{dx}{dy}\right)^2}\, dy = \int_{-2}^{2} \sqrt{1 + (2y)^2}\, dy$$

7. [10] Evaluate the derivative $\dfrac{d}{dx} \displaystyle\int_3^x \frac{1}{1+t^5}\, dt$. (Hint: do not try to evaluate the integral.)

The inverse FTC says that the derivative of the integral is the integrand when the variable is the upper limit of integration, that is, $\dfrac{d}{dx} \displaystyle\int_a^x f(t)\, dt = f(x)$ Therefore, this derivative is $\dfrac{1}{1+x^5}$.