Math 121 Calculus II

First Test Answers
February 2016

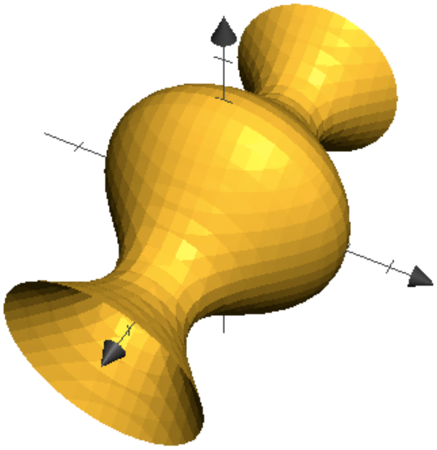
Scale. A+ 99–104. A 92–97. A- 86–91. B+ 81–85. B 75–80. B- 70–74. C+ 65–69. C 60–64. C- 55–59. D+ 50–54. D 35–49. Median 72.

The point values on the original test problems and the summary chart didn't match exactly. The corrected point values are on this answer sheet.

Extra problems. If you didn't do as well on as you would have liked, you may do the on-line extra problems (MyMathLab) by Wednesday. The extra problems are optional. If you get them all correct, I'll adjust your score up by half the difference, 100 minus your test score, so if you're test score is 60, I'll adjust it to 80. If you don't get them all correct, I'll prorate it, so if your score is 60 and you get half the extra problems correct, I'll adjust your score to 70.

When you're doing these extra problems, you may consult the textbook and your class notes, and you may use a calculator. Do not, however, get help from anyone else when doing them.

1. [13] **On areas of surfaces of revolution.** The curve $y = 2 + \cos x$, for x in the interval $[-3, 3]$, is rotated around the x-axis to generate a surface of revolution. Write down an integral which gives the area of this wavy surface. Do not evaluate the integral.



In general, the surface of a surface of revolution is given by the integral

$$\int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

when the axis of revolution is the x-axis. In this example, $\dfrac{dy}{dx} = -\sin x$, so the area is

$$\int_{-3}^{3} 2\pi(2 + \cos x)\sqrt{1 + \sin^2 x}\, dx.$$

2. [15] **On exponential functions.** A colony of bacteria is grown under ideal conditions in a laboratory so that the population increases exponentially with time. Initially there were 1000 bacteria, and after 4 hours there are 4000 bacteria. How many bacteria were there after 3 hours?

There are a couple of ways to arrive at the answer. Here's one. We know that the function giving the number of bacteria at time t is of the form $f(t) = Ae^{kt}$ where $A = 1000$ is the initial population. We can use the fact that $f(4) = 4000$ to determine k as follows. $4000 = f(4) = 1000e^{k4}$, so $4 = e^{4k}$. Therefore $\ln 4 = 4k$, and $k = \frac{1}{4}\ln 4$. Therefore

$$f(t) = 4000e^{(t \ln 4)/4}$$

If you like, you could simplify that to $f(t) = 4000 \cdot 4^{t/4}$, but that simplification is not necessary.

So $f(3) = 4000e^{(3 \ln 4)/4}$, or more simply $4000 \cdot 4^{3/4}$.

3. [15] **On differential equations.** Use the method of separation of variables to solve the differential equation

$$\frac{dy}{dx} = x(1 + y^2).$$

Your answer should be in the form $y = f(x)$ where the function f involves a constant C of integration.

Separating the variables gives the equation

$$\frac{1}{1+y^2}\, dy = x\, dx,$$

and integrating that gives

$$\int \frac{1}{1+y^2}\, dy = \int x\, dx.$$

The left integral is $\arctan y$ while the right one is $\frac{1}{2}x^2$. Therefore

$$\arctan y = \tfrac{1}{2}x^2 + C.$$

Hence,

$$y = \tan(\tfrac{1}{2}x^2 + C).$$

4. [30; 10 points each part] **On integration.** Evaluate the following integrals. Show your work for credit.

a. $\displaystyle\int \theta \cos\theta\, d\theta$

Integration by parts works here with $u = \theta$ and $dv = \cos\theta\, d\theta$. Then $du = d\theta$ and $v = \sin\theta$. Therefore the integral equals

$$\theta\sin\theta - \int \sin\theta\, d\theta = \theta\sin\theta + \cos\theta + C$$

b. $\int \arctan x\, dx$

Use integration by parts with $u = \arctan x$ and $dv = dx$. Then $du = \dfrac{1}{1+x^2}\, dx$ and $v = x$. Therefore the integral equals

$$x\arctan x - \int \frac{x}{1+x^2}\, dx = x\arctan x - \tfrac{1}{2}\log(1+x^2) + C$$

(Note that the function $\arctan x$ is *not* equal to $(\tan x)^{-1}$.)

c. $\int \dfrac{\sqrt{x^2-16}}{x^4}\, dx$

There are several ways to evaluate this integral, but the best way is to start by using the trig sub $x = 4\sec\theta$, $dx = 4\sec\theta\tan\theta\, d\theta$, $\sqrt{x^2-16} = 4\tan\theta$. The integral then becomes

$$\int \frac{4\tan\theta}{(4\sec\theta)^4} 4\sec\theta\tan\theta\, d\theta = \tfrac{1}{16}\int \frac{\tan^2\theta}{\sec^3\theta}\, d\theta$$

At this point there are several ways to proceed. One of them is to convert the tangents and secants to sines and cosines. Then the integral can be written as

$$\tfrac{1}{16}\int \sin^2\theta\cos\theta\, d\theta$$

This is one of the integrals we studied in the section on products of powers of trig functions. The substitution $u = \sin\theta$ quickly gives the antiderivative

$$\tfrac{1}{48}\sin^3\theta + C$$

One way to convert this back to the original variable x is to use the triangle which tells you that $\sin\theta = \dfrac{\sqrt{x^2-16}}{x}$. You could also use the identity $\sin\theta = \dfrac{\tan\theta}{\sec\theta}$. Therefore, the integral is equal to

$$\frac{(x^2-16)^{3/2}}{48x^3} + C.$$

5. [18] **On partial fractions.**

a. The rational function $\dfrac{5x-4}{(x+1)(x-2)}$ is equal to the sum of the partial fractions $\dfrac{A}{x+1} + \dfrac{B}{x-2}$. Determine the values of A and B.

Clear the denominators to get

$$5x - 4 = A(x-2) + B(x+1).$$

Setting $x = 2$ gives $6 = B2$, while setting $x = -1$ gives $-9 = A(-3)$. Therefore, $A = 3$ and $B = 2$

b. Use your answer in part a to evaluate the integral

$$\int \frac{5x-4}{(x+1)(x-2)}\, dx = 2\ln|x+1| + 3\ln|x-2| + C$$

6. [15] **On Improper integrals** Use the fact that $\int xe^{-x}\, dx = -(x+1)e^{-x} + C$ to evaluate the improper integral $\int_1^\infty xe^{-x}\, dx$. Explain in a sentence how you used limits to find your answer.

The improper integral is equal to a limit, namely

$$\begin{aligned} & \lim_{b\to\infty} \int_1^b xe^{-x}\, dx \\ = \quad & \lim_{b\to\infty} -(x+1)e^{-x}\Big|_1^b \\ = \quad & \lim_{b\to\infty} \left((-(b+1)e^{-b}) - (-2e^{-1})\right) \\ = \quad & 2e^{-1} - \lim_{b\to\infty}(b+1)e^{-b} \end{aligned}$$

This limit is of the indeterminate form $\infty \cdot 0$. To evaluate that write it as a quotient and use L'Hôpital's rule.

$$\lim_{b\to\infty}(b+1)e^{-b} = \lim_{b\to\infty}\frac{b+1}{b^x}$$

Since this limit is now of the form $\dfrac{\infty}{\infty}$, we can apply L'Hôpital's rule to see that it is equal to

$$\lim_{b\to\infty}\frac{1}{e^b} = 0.$$

Thus, the value of the integral is $2e^{-1} - 0 = \dfrac{2}{e}$.

Incidentally, the graph of $y = xe^{-x}$ looks like