

Origin of the Fundamental Theorem of Calculus  
 Math 121 Calculus II  
 Spring 2015

Calculus has a long history. Although Newton and Leibniz are credited with the invention of calculus in the late 1600s, almost all the basic results predate them. One of the most important is what is now called the Fundamental Theorem of Calculus (FTC), which relates derivatives to integrals.

**Uniform motion.** Uniformly changing quantities were well understood by the ancient Greek mathematicians. Autolycus of Pitane (ca. 360 BCE–ca. 290 BCE) studied the circular uniform motion of stars in the heavens. He said an object moved uniformly if it traversed equal distances in equal times. That means that for each fixed time period, the object moves the same distance.

Archimedes (ca. 287 BCE–ca. 212 BCE) used this definition to prove that an object moving uniformly traverses distances proportional to times. That means that the distance  $x$  traversed in a time interval of length  $t$  is proportional to the distance  $y$  traversed in a time interval of length  $s$ :

$$\frac{x}{y} = \frac{t}{s}.$$

Note that in this statement there is no mention of velocities, only of distances and times. The ancient Greeks accepted ratios of two quantities as long as they were of the same kind. Here we have the ratio of two distances equal to the ratio of two time intervals. But the ancient Greeks did not accept mixed ratios, for instance the ratio of a distance to a time.

Once mixed ratios were accepted, the above proportion could be written in the alternate form

$$\frac{x}{t} = \frac{y}{s}.$$

We interpret that as saying the velocity  $x/t$  during the first time interval equals the velocity  $y/s$  during the second time interval. In other words, an object moves uniformly when it has a constant velocity. The modern formula for uniform motion says that the distance traveled equals the product of the rate (or velocity) and the elapsed time.

**Nonuniform motion.** By the 1300s scholars were comfortable with velocity, the rate of change of a changing quantity. They used the term velocity in a more general way than we do now. We use it as a the rate of change of the position (or distance) of an object that moves over time. For them, the dependent variable didn't have to be position and the independent variable didn't have to be time; nonetheless, for purposes of exposition, let's limit ourselves to an object that moves over time.

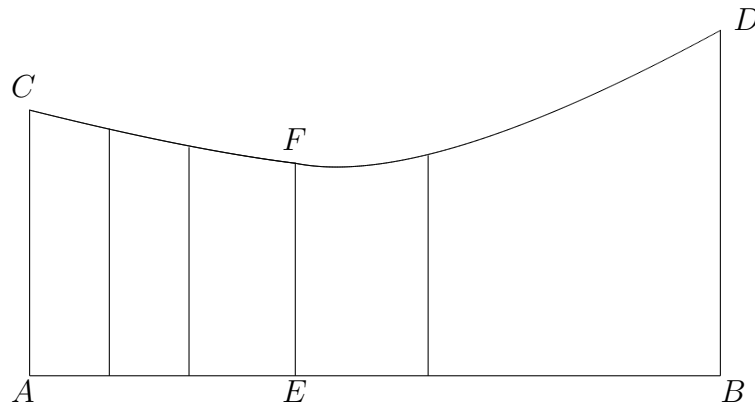
They were trying to understand nonuniform motion, that is, when the velocity is not constant. The problem was that velocity is only defined under uniform motion. What is

velocity if the motion is not uniform? (We've answered that question in terms of limits, but that answer came centuries later.)

Four of these scholars at Merton College in Oxford University—Thomas Bradwardine, William Heytesbury, Richard Swineshead, and John Dumbleton—studied nonuniform motion in the first half of the 1300s. Even though they couldn't precisely define velocity, they worked with velocities as if they were real quantities. Furthermore, they understood when the velocity was changing, it had a rate of change, the acceleration. (Or deceleration if the velocity was decreasing. They didn't know about negative numbers.)

One of their discoveries about a certain nonuniform motion is called the Merton Mean Speed Theorem. It says that if an object is moving with a constant acceleration, then the distance it travels is the same distance it would travel if it were moving at a constant velocity, that velocity being the average of its initial and final velocity. This happens to be the motion of a body in free fall, but there's no indication that these Merton scholars knew that. That result was something Galileo (1564–1642) discovered much later.

**Oresme's Fundamental Theorem of Calculus** Nicole Oresme (ca. 1323–1382) was at the University of Paris and expanded the analytic study of changing quantities. He had a graphical interpretation very similar to the modern graph  $y = f(x)$  of a function in the  $(x, y)$ -plane, although analytic geometry and coordinates were yet to be developed by Fermat and Descartes in the 1600s.



He represented time as a line, much as Aristotle had done long before, so that instants in a time interval were represented by points on a horizontal line segment  $AB$ , which he called the longitude. Given a moving object, at each instant in time  $E$  that moving object has a velocity, and he represented that velocity by a vertical line segment  $EF$  proportional to the velocity; each vertical line segment he called a latitude. These latitudes together formed a plane region  $ABDC$ , which he called a form, bounded on the bottom by the original longitude  $AB$ , on the left by the initial latitude  $AC$  representing the initial velocity, on the right by the final latitude  $BD$  representing the final velocity, and on the top by the curve  $CFD$  which he called the summit curve. He then argued that the area of that form  $ABDC$  is proportional to the distance traveled.

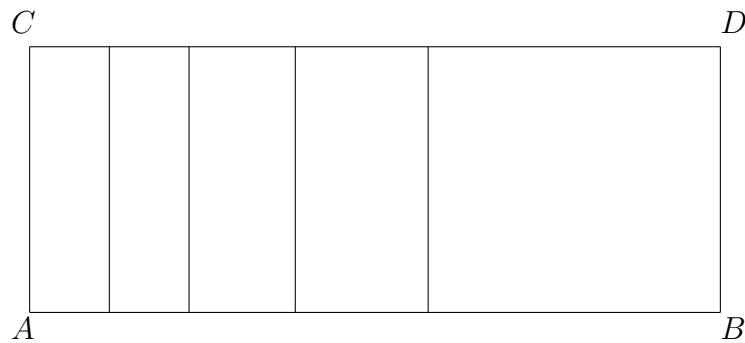
Oresme made the various lengths proportional to distances or times since he thought (as the ancient Greeks did) that they're different kinds of things, but we would probably use the language of equality: we would make length of the longitude equal the elapsed time, and the length of a latitude equal the velocity at that instant.

We can write his result using Leibniz' notation as

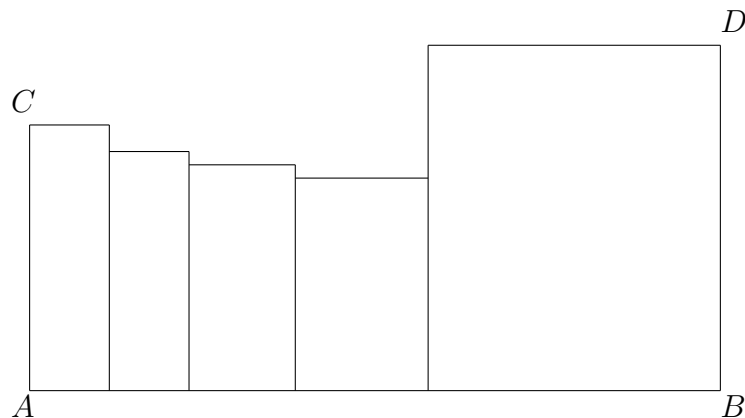
$$\int_a^b f'(x) dx = f(b) - f(a).$$

In the 1300s there was no symbolic algebra at all—no equal sign, no minus sign, and no variables.

Oresme gave examples of this principle and used it to prove the Merton mean speed theorem.



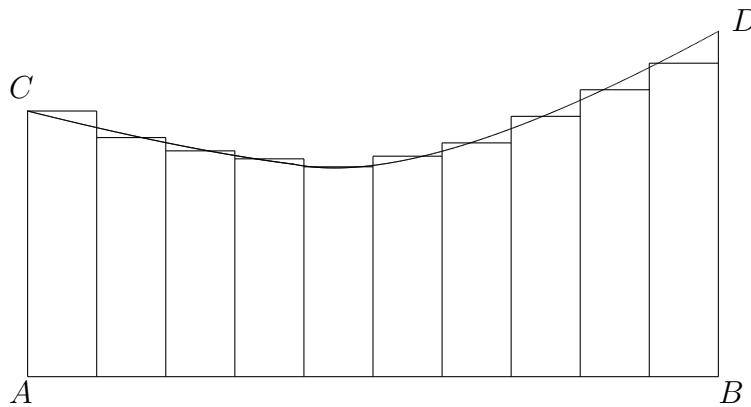
Here's the form for uniform motion. Since the velocity is constant for uniform motion, all the latitudes are equal, so the form has a horizontal summit, that is, the form is a rectangle. The distance traveled is that velocity times the elapsed time, but the area of the rectangle is also the velocity (length of the latitudes) times the elapsed time (length of the longitude).



Next, consider an object moves in a series of uniform motions. Over the first time interval, it has one velocity, over the second time interval a second velocity, and so forth. The form is made of a first rectangle, a second rectangle just to its right, and so forth. The area of the form is the sum of the areas of the rectangles, but the area of each rectangle is the distance traveled over the corresponding time interval, so the area of the form is the total distance traveled.

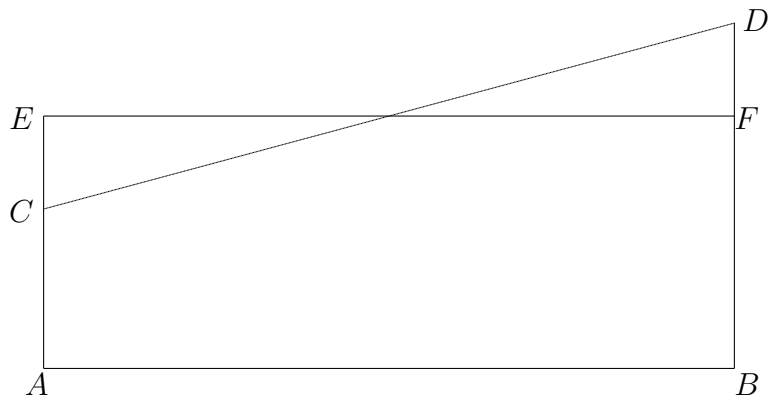
Finally, suppose that the object has a nonuniform motion. The velocity is not constant anywhere, but changes. The form has a curve at the top. There are a couple of ways to be convinced that this principle (FTC) applies in this case also.

One argument is to employ some kind of concept of limiting approximations. Here's one argument of that type. Divide the latitude into small subintervals and assume that the velocity during each subinterval is constant, say the actual velocity at some instant in that subinterval. That gives the motion as described in the previous case, so the total area of the rectangles is the distance traveled for that motion. Since the velocity for that motion is close to the velocity of the original nonuniform motion, therefore the area of the curved form is approximately the distance traveled for that nonuniform motion. Some kind of limiting argument is needed to complete the argument. (Methods of this sort were used later by Fermat, Newton, Riemann, and Darboux.)



An alternate argument for FTC relies on a different interpretation of motion. Aristotle had said that motion takes place over an interval. So velocity is defined over an interval of time rather than at a point in time. One interpretation of that is that even for nonuniform motion, velocity is constant over intervals, although they are very short intervals. Then the motion actual is a series of uniform motions and the previous case already does the general case. (This is very similar to Leibniz' approach to motion where the subintervals of constant velocity are infinitesimals.)

**Oresme's proof of the Merton mean speed theorem.** Oresme used this principle to prove the Merton mean speed theorem. Suppose that an object undergoes constant acceleration. Then its velocity increases (or decreases when the acceleration is negative) at a constant rate. Therefore the summit line of the form is a slanted straight line. (The complete argument that it's a straight line involves similar triangles.) Thus, the form is a trapezoid  $ABDC$  with vertical parallel sides  $AC$  and  $BD$ .



But the area of this trapezoid  $ABDC$  is the same as the area of a rectangle  $ABFE$  on the same base but whose height  $AE = BF$  is the average height of the left and right sides of the trapezoid. Thus the distance traveled under constant acceleration is the same as the distance traveled by an object going a constant velocity, that being the average of the initial and final velocities.

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