

Antiderivatives
Math 121 Calculus II
Spring 2015

Antiderivatives and the constant of integration. We'll start out this semester talking about antiderivatives. If the derivative of a function F is f , that is, $F' = f$, then we say F is an *antiderivative* of f .

Of course, antiderivatives are important in solving problems when you know a derivative but not the function, but we'll soon see that lots of questions involving areas, volumes, and other things also come down to finding antiderivatives. That connection we'll see soon when we study the Fundamental Theorem of Calculus (FTC). For now, we'll just stick to the basic concept of antiderivatives.

We can find antiderivatives of polynomials pretty easily. Suppose that we want to find an antiderivative $F(x)$ of the polynomial

$$f(x) = 4x^3 + 5x^2 - 3x + 8.$$

We can find one by finding an antiderivative of each term and adding the results together. Since the derivative of x^4 is $4x^3$, therefore an antiderivative of $4x^3$ is x^4 . It's not much harder to find an antiderivative of $5x^2$. Since the derivative of x^3 is $3x^2$, an antiderivative of $5x^2$ is $\frac{5}{3}x^3$. Continue on and soon you see that an antiderivative of $f(x)$ is

$$F(x) = x^4 + \frac{5}{3}x^3 - \frac{3}{2}x^2 + 8x.$$

There are, however, other antiderivatives of $f(x)$. Since the derivative of a constant is 0, we can add any constant to $F(x)$ to find another antiderivative. Thus, $x^4 + \frac{5}{3}x^3 - \frac{3}{2}x^2 + 8x + 7$ is another antiderivative of $f(x)$. If C is any constant, then $x^4 + \frac{5}{3}x^3 - \frac{3}{2}x^2 + 8x + C$ is an antiderivative of $f(x)$. We'll usually use the letter C to indicate this arbitrary constant that appears when we antidifferentiate. We'll call it the *constant of integration*.

Is the constant of integration always constant? Are there any other antiderivatives of f besides these? No. And why not? Suppose that we had two different antiderivatives of f , call them F and G . Then both $F' = f$ and $G' = f$, so $F' - G' = 0$. Since the difference of two derivatives equals the derivative of the difference, therefore $(F - G)' = 0$. Thus, the derivative of the function $F - G$ is 0. We know a theorem that applies in this case, and it says that $F - G$ is constant. Thus, the two different antiderivatives of f differ by a constant.

Here's the theorem, one that we studied in the first semester of calculus. It was a corollary of the Mean Value Theorem (MVT).

Theorem. If the derivative of a function is 0 on an interval, then the function is constant on that interval.

Warning. That theorem doesn't imply that the antiderivatives of every function differ by a constant; only those functions defined on an interval

Here are two antiderivatives of the function $f(x) = 1/x$. Note that this function is not defined at 0, but is defined on two separate intervals, namely, the two open intervals $(-\infty, 0)$ and $(0, \infty)$. One antiderivative is the function $F(x) = \ln |x|$. Another is defined by cases:

$$G(x) = \begin{cases} 1 + \ln |x| & \text{if } x > 0 \\ 2 + \ln |x| & \text{if } x < 0 \end{cases}$$

These two antiderivatives, F and G , do not differ by a constant. For positive x they differ by the constant 1, that is $G(x) - F(x) = 1$; but for negative x they differ by the constant 2, that is, $G(x) - F(x) = 2$. There is not one constant that works for all values of x .

That example notwithstanding, we'll still use the constant of integration C when we find antiderivatives. We'll say the the general antiderivative of $1/x$ is $\ln |x| + C$, but keep in mind that different constants may be needed on the different intervals of definition of the functions.

Do all functions have antiderivatives? All polynomials do and lots of other functions do. Indeed, all continuous functions have antiderivatives. But noncontinuous functions don't. Take, for instance, this function defined by cases.

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

You can find an antiderivative of f for negative x and for positive x , namely

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } x > 0 \end{cases}$$

but there's no way to define $F(0)$ to make F differentiable at 0 (since the left derivative at 0 is 0, but the right derivative at 0 is 1).

We'll see soon that continuous functions do have antiderivatives when we look the inverse of the Fundamental Theorem of Calculus.

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