Summary of foundations of integration
Math 121 Calculus II
Spring 2015

For the first few meetings of this course in integration we will discuss the history and foundations of integral calculus as the mechanics of finding antiderivatives and evaluating definite integrals.

The fundamental theorem of calculus (FTC). Historically, calculus began with Oresme's discovery of the fundamental theorem of calculus. Nicole Oresme (ca. 1323–1382) expounded a graphical analysis of changing quantities. I'll use modern notation to summarize his results. If a dependent quantity $y = f(x)$ is changing as an independent quantity x changes as x varies from $x = a$ to $x = b$, Oresme drew what we would call the graph of the derivative $y = f'(x)$ and argued that the area under that curve, above the x-axis, and between the lines $x = a$ and $x = b$ was equal to the total change in y, that is, $f(b) - f(a)$. This area was later called the integral of f' from a to b, and denoted $\displaystyle\int_a^b f'(x)\,dx$. Oresme's result, now called the fundamental theorem of calculus, can be written simply as

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

Oresme's language was completely geometric as symbolic algebra had not yet been invented. Neither had analytic geometry, so he did not refer to an x-axis, but simply a straight line to represent an interval, and that interval was usually a time interval since his examples usually took time as the independent variable. At each instant of time, he erected a perpendicular line for the velocity at that time, what we now call the derivative. He took the instantaneous velocity as an understood concept, and with just a few properties of velocity, he could prove his theorem.

We'll prove the FTC using modern notation and Riemann's 19th century definition of integrals which is not that different from Oresme's.

Integration in the 1600s. Bonaventura Cavalieri (ca. 1598–1647) wrote the first textbook that included some properties of integrals. He argued what is now called Cavalieri's principle: if two regions are bounded between two parallel lines, and their parallel cross sections are all proportional, then the areas of the regions are in the same proportion. It follows from this principle that integration is linear, which in modern notation says that

$$\begin{aligned}
\int_a^b (f+g)(x)\,dx &= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \\
\int_a^b (f-g)(x)\,dx &= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \\
\int_a^b cf(x)\,dx &= c\int_a^b f(x)\,dx
\end{aligned}$$

where c is a constant. He also proved that if n is a positive integer,

$$\int_0^b x^n \, dx = \frac{b^{n+1}}{n+1}.$$

Others in the period determined other integrals. Fermat and Torricelli, about 1640, determined the integral of $x^{m/n}$ for rational numbers m/n other than -1. Fermat's argument was a rigorous proof based on rectangular overestimates and underestimates of these areas. An integral defined in terms of rectangular overestimates and underestimates is sometimes called a Darboux integral.

Another obvious property of integrals is that integrals are additive on intervals, that is,

$$\int_a^c f(x) \, dx = \int_a^b f(x) \, dx + \int_b^c f(x) \, dx.$$

The inverse of the fundamental theorem of calculus. In some texts the FTC has two parts—the one mentioned above and its inverse. The inverse, FTC^{-1}, says that given an integrable function $f(x)$, if you define a new function $F(x)$ in terms of it as

$$F(x) = \int_a^x f(t) \, dt$$

where a is any constant, then the derivative of F is f. In other words, the derivative F' of the integral F is the original function f.

This theorem FTC^{-1} can be thought of as the inverse to FTC when we just take the special case of $\text{FTC}_{f(a)=0}$ where $f(a) = 0$. Then $\text{FTC}_{f(a)=0}$ says $\int_a^b f'(x) \, dx = f(b)$, that is, the integral $F(b) = \int_a^b f'(x) \, dx$ of the derivative $f'(x)$ is the original function f. This function $F(x)$ is sometimes called an *accumulation* function since as x increases from a to b, the vertical line at x sweeps right accumulating more of the region under the curve, and $F(x)$ indicates the area of that region.

Together, these two theorems say that the two processes of integration and differentiation are inverse to each other.

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