CLARK UNIVERSITY 1887

How to evaluate an integral
Math 121 Calculus II
Spring 2015

By now you know integration is much more difficult than differentiation. There are rules for differentiation that allow you to differentiate any thing you can write down. There aren't for integration. You know the rules for integration that involve sums, differences, constants, and powers of x, but there are no rules for products, quotients, or compositions. To make up for the lack of those rules, we have a series of techniques that sometimes help and sometimes don't, but even when they do help, sometimes they're complicated to carry out.

The two most important techniques are substitution and integration by parts. Then there are numerous special techniques that help integrate certain special forms of functions. We've looked at integrating products of trig functions, trigonometric substitutions, partial fractions, and using tables. Remember that it's always fair to use previous results. So, for instance, since we've found $\int \sin^2 x\, dx$ once, we don't have to do it again. In fact, the tables of integrals are nothing but summaries of integrals we have found, or could find, by the other techniques. There are plenty more techniques for special integrals, but the ones we've studied are the most important, and they're quite enough for an introductory course. The problem is deciding what technique to use on a particular integral. Here are some guidelines.

1). If you see a composition somewhere in the integral, then it may help to substitute for the inner function. For instance, if $\sin\sqrt{x}$ appears in the integral, maybe the substitution $u = \sqrt{x}$ will help. In order for the substitution to actually work, however, the derivative $\frac{du}{dx}$ has to appear as a factor of the integrand. (Section 5.7.)

1′). If you have a quotient for an integrand, and the derivative of the denominator is the numerator or a factor of the numerator, then, by all means, use the substitution where you let u be the denominator. (Section 5.7.)

2). If the integrand is a product, then maybe integration by parts $\int u\, dv = uv - \int v\, du$ will help. When you select u and dv, try to choose u so that its derivative is simpler than it is, and dv so that you can integrate it. (Section 8.2.)

2′). Even when the integrand isn't a product, integration by parts might help if the derivative of the integrand is simpler. Try letting u be the whole integrand, and take dv to be dx. (Section 8.2.)

3). In the special case that the integrand is a power or product of powers of trig functions, you know techniques for that. (Section 8.3.) Actually, there are many cases. Some depend on the double angle formulas for cosine, others on integration by parts.

4). When you spy a square root of sums or differences of squares, $\sqrt{\pm x^2 \pm a^2}$, somewhere in the integrand then a trig sub should help (Section 8.4.) There are three kinds of trig subs depending on form of the expression in the integrand: for $\sqrt{a^2 - x^2}$ use $x = a\sin\theta$, for $\sqrt{a^2 + x^2}$ use $x = a\tan\theta$, or for $\sqrt{x^2 - a^2}$ use $x = a\sec\theta$.

5). If the integrand is a rational function, then the method of partial fractions will work, but it's a lot of work. (Section 8.5.) There are four steps.

Step 1: Make sure the numerator of the rational function has a lower degree than the denominator by dividing the denominator into the numerator using long division.

Step 2: Factor the denominator.

Step 3: Write the rational function as a sum of partial fractions. This involves the 'method of undetermined coefficients' and solving a set of simultaneous equations.

Step 4: Integrate the partial fractions. If their denominators are linear, then it's easy and you'll get some natural logs; but if they're irreducible quadratics, then it will be more difficult and the answer will involve arctangents.

The goal of much of mathematics is to reduce difficult tasks to straightforward algorithms. That is possible in some parts of mathematics, for instance in algebra and differential calculus. Algorithms in algebra can solve linear and quadratic equations. We have enough rules for differentiation so that differential calculus is reduced to an algorithm.

Unfortunately, integral calculus doesn't have enough rules, so it can't be reduced to an algorithm. So we move to the second best solution, and that involves a series of techniques that help us to integrate the more important functions.

Even so, some functions, although integrable (because they're continuous) cannot be found as elementary functions (expressed in terms of trig functions, inverse trig functions, logs, and exponentiation). Even a simple-looking integral such as $\int \sqrt{1+x^3}\, dx$ is not an elementary function.

Math 121 Home Page at `http://math.clarku.edu/~ma121/`