The Logistic Population Model
Math 121 Calculus II
Spring 2015

Summary of the exponential model. Back a while ago we discussed the exponential population model. For that model, it is assumed that the rate of change $\frac{dy}{dt}$ of the population y is proportional to the current population. If r is the constant of proportionality, that's the exponential differential equation

$$\boxed{\frac{dy}{dt} = ry}$$

and that has the general solution

$$\boxed{y = Ae^{rt}}$$

where A is the initial population $y(0)$. Rather than using the base e for exponentiation, any other convenient base b can be used

$$y = Ab^{st}$$

where $s = \frac{r}{\ln b}$.

There are many assumptions of this exponential model. In particular, it is assumed that there are unlimited resources and there is no competition within the population.

More reasonable models for population growth can be devised to fit actual populations better at the expense of complicating the model.

The logistic model. Verhulst proposed a model, called the logistic model, for population growth in 1838. It does not assume unlimited resources. Instead, it assumes there is a *carrying capacity* K for the population. This carrying capacity is the stable population level. If the population is above K, then the population will decrease, but if below, then it will increase.

For this model it is assumed that the rate of change $\frac{dy}{dt}$ of the population y is proportional to the product of the current population y and $K - y$, or what is the same thing, proportion to the product $y(1 - y/K)$. That gives us the logistic differential equation

$$\frac{dy}{dt} = ry(1 - y/K).$$

Here, r is a positive constant. Note that when $y < K$,$\frac{dy}{dt}$ is positive, so y increases, but when $y < K$, the derivative is negative, so y decreases.

We can solve this differential equation by the method of separation of variables. First, separate the variables to get

$$\frac{1}{y(1 - y/K)}\, dy = r\, dt$$

and integrate

$$\int \frac{1}{y(1 - y/K)}\, dy = \int r\, dt.$$

Of course, $\int r\, dt = rt + C$, but what about the integral on the left side of the equation?

For that, we'll need the method of partial fractions. Write $\frac{1}{y(1 - y/K)}$ as the sum of two simpler rational functions:

$$\frac{1}{y(1 - y/K)} = \frac{A}{y} + \frac{B}{1 - y/K}$$

where A and B are coefficients yet to be determined. Clear the denominators to get the equation

$$1 = A(1 - y/K) + By = A - \frac{A}{K}y + By$$

from which it follows that $A = 1$ and $B = 1/K$. Thus,

$$\frac{1}{y(1 - y/K)} = \frac{1}{y} + \frac{1/K}{1 - y/K} = \frac{1}{y} + \frac{1}{K - y}.$$

Therefore,

$$\begin{aligned}\int \frac{dy}{y(1-y/K)} &= \int \frac{dy}{y} + \int \frac{dy}{K-y} \\ &= \ln y - \ln|K-y| \\ &= \ln\left|\frac{y}{y-K}\right|.\end{aligned}$$

Now that we've integrated the left side of the equation, we can continue

$$\ln\left|\frac{y}{y-K}\right| = rt + C$$

Next, the application of a little algebra gives us

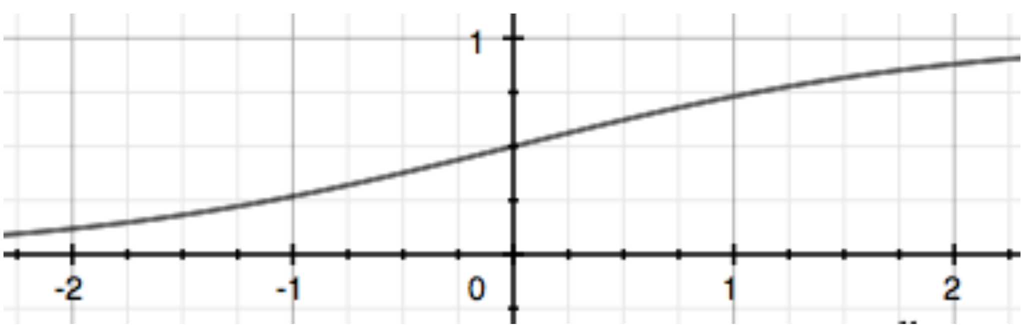
$$y = \frac{K}{1+Ae^{-rt}}$$

where A is a constant.

The graph of this function is asymptotic to the y-axis on the left, asymptotic to the line $y = K$ on the right, and symmetric with respect to the point where $y = K/2$, which is an inflection point. To its left, the graph is concave upward, but to its right, concave downward.

Here's the graph of for $K = 1$, $A = 1$, and $r = 1$, so that

$$y = \frac{1}{1+e^{-t}}.$$



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