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Permutations and determinants
Math 130 Linear Algebra
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One way to construct determinants is in terms of permutations. That construction depends on a particular property of permutations, namely, their parity.

Permutations. A *permutation* of the set $\{1, 2, 3, \ldots, n\}$ is a list of its n elements where each element appears exactly once in the list. For instance, the six permutations of the set $\{1, 2, 3\}$ are 123, 132, 213, 231, 312, and 321. In general, there are $n!$ permutations of a set of size n. This expression $n!$, read n factorial, is the product of the integers from 1 through n.

A permutation σ is often written as a list, such as $\sigma = 24153$, but you can also treat it as a function on the set. This permutation σ is a function on the set $\{1, 2, 3, 4, 5\}$, where σ sends 1 to 2, 2 to 4, 3 to 1, 4 to 5, and 5 to 3. We'll write $\sigma_1 = 2$, $\sigma_2 = 4$, $\sigma_3 = 1$, $\sigma_4 = 5$, and $\sigma_5 = 3$.

Permutation matrices. One way to look at a permutation is to treat it as a matrix itself. First, think of the permutation as an operation rather than a list. For instance, associate to the permutation $\sigma = 24153$ the following 5×5 matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

This matrix then operates on a column vector as a permutation

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} a_2 \\ a_4 \\ a_1 \\ a_5 \\ a_3 \end{bmatrix}$$

Thus, the permutation matrix permutes the rows of another matrix. The row 1 is replaced by row 2, row 2 by row 1, row 3 by row 4, row 4 by row 5, and row 5 by row 3.

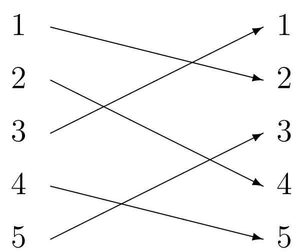
Preview of permutations and determinants. When we construct the determinant of a square $n \times n$ matrix, which we'll do in a moment, it will be defined as a sum/difference of $n!$ terms, each term being a product of n elements, one element chosen out of each row and column.

Our first question is: why are there $n!$ ways to choose one element out of each row and column? Each choice is determined by which column to choose for each row. So, if the element $a_{1\sigma_1}$ is chosen for the first row, $a_{2\sigma_2}$ is chosen for the second row, ..., and $a_{n\sigma_n}$ is chosen for the nth row, that choice is determined by the permutation $\sigma = \sigma_1\sigma_2 \ldots \sigma_n$, that is, a permutation of the set $\{1, 2, \ldots, n\}$. So, there are $n!$ choices, and each corresponds to a permutation.

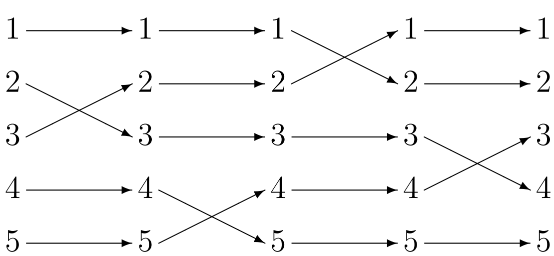
We'll form all $n!$ products of n elements, one element chosen out of each row and column. Half of these $n!$ we'll add, the other half we'll subtract. The result will be the determinant. We'll add those that correspond to "even permutations" and subtract those that correspond to "odd permutations". Before we can do that, we'll have to define what it means for a permutation to be even or odd.

Parity of a permutation. A *transposition* is a particularly simple permutation. It exchanges exchanges exactly two elements and leaves all the others fixed. Some transpositions of the set of 5 elements, for example, are 21345, 12543, and 52341, but the permutation 23145 isn't a transposition since 1, 2, and 3 are all moved.

Each permutation can be expressed as a product of transpositions in many ways. One nice way to visualize a permutation is by drawing lines connecting the initial list of numbers from 1 through n to their final positions. For example, the permutation $\sigma = 24153$ can be drawn as



In this figure, there are 4 arrow crossings. Each one can be interpreted as a transposition, and you can use that interpretation as a way to represent the permutation as a composition of transpositions. Here it's shown as the product of these four transpositions: (23), (45), (12), and (34).



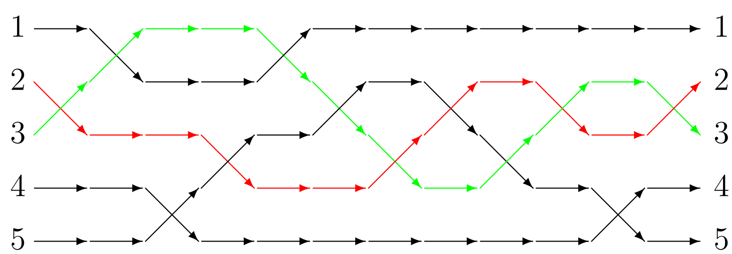
There are many other ways that this permutation can be represented as composition of transpositions. Just take any drawing where you draw curves going to the right from the numbers 1, 2, 3, 4, and 5 to their correct targets. (You'll need to make sure that all the crossings are only double crossings and that there are only finitely many of crossings.) The number of crossings will vary, but we'll the parity of the number of crossings doesn't change. We expressed this example as a composition of 4 transpositions, so however it's expressed, it will have to involve an even number of transpositions.

Theorem 1. A permutation cannot be expressed as both the composition of an even number of transpositions and an odd number of transpositions.

Proof. Suppose a permutation σ can be expressed as a composition of transpositions in two ways $\sigma = \tau_1 \cdots \tau_m = \rho_1 \cdots \rho_n$ where each τ_i and ρ_i is a transposition. Then $\sigma^{-1} = (\rho_1 \cdots \rho_n)^{-1} = \rho_n^{-1} \cdots \rho_1^{-1}$, and since the inverse of any transposition is itself, therefore $\sigma^{-1} = \rho_n \cdots \rho_1$. Therefore the identity permutation is the product of $m + n$ transpositions, $\tau_1 \cdots \tau_m \rho_n \cdots \rho_1$. In the following lemma, we'll show that that identity permutation can only be expressed as a composition of an even number of transpositions. That will imply that $m+n$ is even. Hence m and n have the same parity. Q.E.D.

Lemma 2. The identity permutation can only be expressed as the composition of an even number of transpositions.

Proof. Consider a diagram that expresses the identity as a composition of transpositions.



Each strand in the diagram starts at one number and ends at the same number.

Consider the strands labelled 2 and 3 in the diagram, drawn in red and green, respectively. The red strand starts out above the green. Each time the crosses the green strand, it switches from above to below, or from below to above. Since it ends up above the green where it started, there have to be an even number of times that the red strand crosses the green strand.

Likewise, for any other pair of numbers i and j, the number of times the i strand crosses the j strand has to be even.

Therefore, the total number of crossings has to be even. That says that the identity transformation can only be expressed as the composition of an even number of transpositions. Q.E.D.

We can now define the *parity* of a permutation σ to be either *even* if its the product of an even number of transpositions or *odd* if its the product of an odd number of transpositions. The *sign* of σ, denoted $\operatorname{sgn}\sigma$, is defined to be 1 if σ is an even permutation, and -1 if σ is an odd permutation.

Construction of the determinant. The determinant of a square $n \times n$ matrix A is sum of $n!$ terms, one for each permutation σ of the set $\{1, \dots n\}$, where each term is

$$\operatorname{sgn}\sigma\, A_{1\sigma_1} \cdots A_{n\sigma_n}$$

The entries $A_{1\sigma_1}, \dots, A_{n\sigma_n}$ are one taken one from each row i and each column σ_i, and $\operatorname{sgn}\sigma$ is the sign of the permutation σ. Symbolically,

$$|A| = \sum_\sigma \operatorname{sgn}\sigma A_{1\sigma_1} \cdots A_{n\sigma_n}$$

This definition usually is used to compute determinants when n is small, 2 or 3, and it agrees with what we did above. But when n is 4 or greater, there are so many terms that it isn't practical to use the definition to compute the value of a determinant. There are much faster ways of computing determinants.

The determinant of a 4×4 matrix. Let's take a generic matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Look at all $4! = 24$ permutations of the set $\{1, 2, 3, 4\}$ and their parities. Even parities are indicated with $+$, odd with $-$.

1234	$+$	2134	$-$	3124	$+$	4123	$-$
1243	$-$	2143	$+$	3142	$-$	4132	$+$
1324	$-$	2314	$+$	3214	$-$	4213	$+$
1342	$+$	2341	$-$	3241	$+$	4231	$-$
1423	$+$	2413	$-$	3412	$+$	4312	$-$
1432	$-$	2431	$+$	3421	$-$	4321	$+$

Each entry in the above table gives one term in the determinant of A. Thus, reading down the first column, we see that the determinant starts out with the following six terms:

$$+a_{11}a_{22}a_{33}a_{44} - a_{11}a_{22}a_{34}a_{43} - a_{11}a_{23}a_{32}a_{44}$$
$$+a_{11}a_{23}a_{34}a_{42} + a_{11}a_{24}a_{32}a_{43} - a_{11}a_{24}a_{33}a_{42}$$

But besides these, there are 18 more terms.

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